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2.26 Triple-Free Set Constants

A set S of positive integers is called **double-free** if, for any integer x, the set $\{x, 2x\} \not\subseteq S$. Equivalently, S is double-free if $x \in S$ implies $2x \notin S$. Consider the function

$$r(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is double-free}\},$$

that is, the maximum cardinality of double-free sets with no element exceeding n. It is not difficult to prove that

$$\lim_{n\to\infty} \frac{r(n)}{n} = \frac{2}{3};$$

that is, the asymptotic maximal density of double-free sets is $2/3$. Wang [1] obtained both recursive and closed-form expressions for $r(n)$ and, moreover, demonstrated that $r(n) = 2n/3 + O(\ln(n))$ as $n \to \infty$.

Let us now discuss a much harder problem. Define a set S of positive integers to be

- **weakly triple-free** (or **triple-free**) if, for any integer x, the set $\{x, 2x, 3x\} \not\subseteq S$, and
- **strongly triple-free** if $x \in S$ implies $2x \notin S$ and $3x \notin S$.

Unlike the double-free case, the weak and strong senses of triple-free do not coincide. Consider the functions

$$p(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is weakly triple-free}\},$$
$$q(n) = \max\{|S| : S \subseteq \{1, 2, \ldots, n\} \text{ is strongly triple-free}\}.$$

We wish to calculate the constants

$$\lambda = \lim_{n\to\infty} \frac{p(n)}{n}, \quad \mu = \lim_{n\to\infty} \frac{q(n)}{n}.$$

Define an infinite set

$$A = \left\{2^i 3^j : i, j \geq 0\right\} = \{a_1 < a_2 < a_3 < \ldots\}$$
$$= \{1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, \ldots\}$$

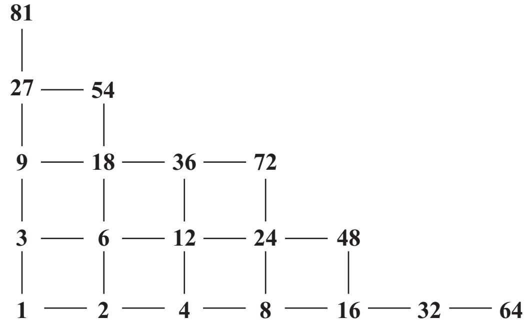

Figure 2.5. Grid graph associated with A_{19}, for which $g_{19} = 10$, $A_{19,0} = \{1, 4, 6, 9, 16, 24, 36, 54, 64, 81\}$, $f_{19} = 6 = h_{19}$, $B_{19,0} = \{1, 6, 8, 27, 36, 48, 64\}$, and $\tilde{B}_{19,0} = \{64\}$.

and A_n to be the first n terms of A; then λ and μ can be written as

$$\lambda = \frac{1}{3}\sum_{n=1}^{\infty}(n - f_n)\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right), \quad \mu = \frac{1}{3}\sum_{n=1}^{\infty} g_n\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right),$$

where the integer sequences

$$\{f_n\} = \{0, 0, 1, 1, 1, 1, 2, 2, 2, 3, 3, 4, 4, 4, 4, 5, 5, 5, 6, 6, 7, 7, 7, 8, 8, \ldots\},$$
$$\{g_n\} = \{1, 1, 2, 2, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 9, 9, 10, 11, \ldots\}$$

will be defined momentarily.

The constant μ has not attracted as much attention as λ. Eppstein [2] showed that g_n is the size of the largest set of nonadjacent vertices in the grid graph A_n (called an **independence number**). For each $k = 0, 1$, define $A_{n,k} \subseteq A_n$ to consist of all elements $2^i 3^j$ satisfying $i + j \equiv k \mod 2$. Then $\{A_{n,0}, A_{n,1}\}$ is a partition of A_n and at least one of these is a maximal independent set, as found by Cassaigne [3]. (See Figure 2.5). From here, Zimmermann [3] computed the **triple-free set constant** to be $\mu = 0.6134752692\ldots$.

By way of contrast, the constant λ has intrigued people for over twenty-five years [4]. Graham, Spencer & Witsenhausen [5] were concerned with general conditions on sets, contained in $\{1, 2, \ldots, n\}$, that avoid the values of linear forms $\sum_{v=1}^{w} c_{uv}x_v$. Among many things, they asked whether λ is irrational. Starting from a table of f_n values in [5], Cassaigne [6] proved that $\lambda \geq 4/5$. Chung, Erdös & Graham [7] showed that f_n is the size of the smallest set of vertices in A_n that intersects every L-shaped vertex configuration of the form $\{2^i 3^j, 2^{i+1}3^j, 2^i 3^{j+1}\} \subseteq A_n$ (called an **L-hitting number**). For each $k = 0, 1, 2$, define $B_{n,k} \subseteq A_n$ to consist of all elements $2^i 3^j$ satisfying $i - j \equiv k \mod 3$. Then $\{B_{n,0}, B_{n,1}, B_{n,2}\}$ is a partition of A_n. Define also $\tilde{B}_{n,k} \subseteq B_{n,k}$ to consist of all elements 2^i, $1 \leq i \equiv k \mod 3$, for which $2^{i-1}3 \notin A_n$. It is known that

$$f_n \leq h_n = \min_{0 \leq k \leq 2} |B_{n,k}| - |\tilde{B}_{n,k}| \leq \left\lfloor \frac{n}{3} \right\rfloor,$$

and consequently $0.800319 < \lambda < 0.800962$. It is conjectured that $f_n = h_n$ for all n, which if true would imply that $\lambda = 0.8003194838\ldots = 1 - 0.1996805161\ldots$.

Given fixed $s > 1$, consider sets S of positive integers for which $\{x, 2x, 3x, \ldots, sx\} \not\subseteq S$ for all integers x. Denote the corresponding asymptotic maximal density by λ_s. What can be said about the asymptotics of λ_s as $s \to \infty$? Spencer & Erdös [8] proved that there exist constants c and C for which

$$1 - \frac{C}{s\ln(s)} < \lambda_s < 1 - \frac{c}{s\ln(s)}$$

for all suitably large s, although specific numerical values were not presented. Also, consider sets T of positive integers for which $\{x, 2x, 3x, 6x\} \not\subseteq T$ for all integers x. The corresponding asymptotic maximal density is exactly $11/12$ [7], which is surprising since the case $s = 3$ was so much more difficult.

More instances of the interplay between the numbers 2 and 3 occur in [2.30.1], which is concerned with powers of $3/2$ modulo 1.

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2.27 Erdös–Lebensold Constant

A strictly increasing sequence of positive integers $a_1, a_2, a_3, \ldots$ is **primitive** [1–3] if $a_i \nmid a_j$ for any $i \neq j$. That is, no term of the sequence divides any other. An example of a finite primitive sequence is the set of all integers m in the interval $\lceil \frac{n+1}{2} \rceil \leq m \leq n$, where n is a positive integer. An example of an infinite primitive sequence consists of all positive integers composed of exactly r prime factors, where r is fixed. We discuss the finite and infinite cases separately. See also [5.5] for a related note.

2.27.1 Finite Case

For each positive integer n, define

$$M(n) = \sup_{\substack{\text{primitive} \\ A \subseteq \{1,2,\ldots,n\}}} \sum_i 1$$